A Note on Stochastic Volatility, GARCH models, and Hyperbolic Distributions*

Stefan R. Jaschke[a]

[a]Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 38, 10117 Berlin, Germany, stefan@jaschke.net

Version 1.1, April 6, 2000

Abstract. We establish a relation between stochastic volatility models and the class of generalized hyperbolic distributions. These distributions have been found to fit exceptionally well to the empirical distribution of stock returns. We review the background of hyperbolic distributions and prove stationary distributions of certain GARCH-type models to be generalized hyperbolic.

Keywords: stochastic volatility, GARCH
JEL Classification: C22
Mathematics Subject Classification (1991): 62P05

1 Introduction

The aim of this paper is to gather some results about the empirical distribution of stock returns and some attempts to match the "heavy tails" of these distributions in stochastic processes models, and to give an introductory overview of the appearance of generalized hyperbolic distributions in this context.

There are some well-known empirical facts about log returns on stocks:

(i) The empirical distribution is leptokurtic compared to the normal distribution.

(ii) Although there is no significant serial correlation in stock returns, there is serial correlation in squared log returns.

*The main part of the study of hyperbolic distributions and the research into stochastic volatility models was done when I was a guest at Aarhus University in 1994. As I continued to work in a different field, the paper never made it into publishable form. During a discussion several years later I was encouraged to somehow complete the idea and make a small note. I like to thank Uwe Küchler, Michael Sørensen, and Rolf Tschernig for helpful comments on this paper. The note was finalized at the Sonderforschungsbereich 373 at Humboldt University Berlin. Financial support by the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

Empirical and theoretical investigations of (i) have a long history. Mandelbrot and Fama proposed *Pareto-stable distributions* to explain the excess kurtosis in stock returns ((Fama; 1965), (Mandelbrot and Taylor; 1967)). Mittnik and Rachev (1993) give an overview and comparison of alternative distributions in modeling stock returns. Evidence against stable Paretian distributions had been accumulated, and Mittnik and Rachev suggest the double Weibull-distribution for stock returns, whose density is

$$f(x;\alpha,\lambda) = \frac{1}{2}\lambda|x|^{\alpha-1}\exp(-\lambda x^{\alpha}) \qquad (\alpha > 0, \lambda > 0).$$

One of their arguments in favor of the Weibull distribution is that tails decrease exponentially. (They estimate α to be close to 1.)

Hyperbolic distributions, which also have exponentially decreasing tails, were independently suggested as distributions of German stock returns by Eberlein and Keller (1994) and Küchler et al. (1994). (The logarithm of the density of a hyperbolic distribution is a hyperbola.) Hyperbolic distributions seem to fit exceptionally well to the return in German stocks represented in the Stock index DAX. Barndorff-Nielsen fitted generalized hyperbolic distributions to Danish stock returns (Barndorff-Nielsen; 1994).

Let $R_k = \log S_k - \log S_{k-1}$ denote the (one-period) log return, given a stock price process $(S_t)_{t\in[0,\infty)}$. Suppose that (R_k) is stationary. We think that the important question now is what kind of stochastic process (R_k) is, rather than looking at its marginal distributions only. Küchler et al. suggest

$$\begin{aligned} dX_t &= m(X_t)dt + \sigma dW_t \\ R_n &= X_n \\ m(x) &= -\frac{\sigma^2}{2}(\alpha\frac{x-\mu}{\sqrt{\delta^2+(x-\mu)^2}} - \beta) \end{aligned}$$

$m(x)$ is chosen in such a way that the stationary distribution of X is hyperbolic. This approach is not satisfying for several reasons. First, a nontrivial $m(x)$ induces an autocorrelated series (R_k) which is not what we observe. Second, in the continuous time version of the model, the stock price would be

$$S_t = \exp\{\int_0^t X_s ds + \mu t\},$$

which is absolutely continuous and of bounded variation. So, there would be arbitrage in the continuous-time option pricing theory (Harrison and Pliska; 1981), if the locally riskless security $S_t^0 = \exp\{\int_0^t r_s ds\}$ wasn't identical to S_t. In this sense, the model proposed in (Küchler et al.; 1994) is not suitable for the valuation of derivative securities.

Barndorff-Nielsen (1994) suggests modeling $\log S_t$ with processes whose increments are independent and generalized hyperbolically distributed. Stock

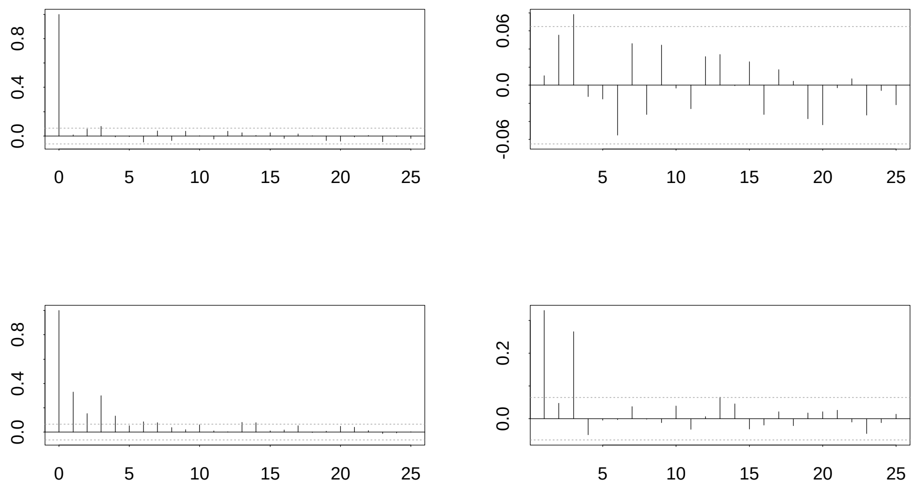

Figure 1: Autocorrelation (left) and partial autocorrelation (right) in weekly log returns in the DAX 1974 – 1992

returns are, however, dependent, although there is no significant linear dependence (serial correlation). The fact, that there is significant serial correlation in squared residuals of many economic time series led to the development of stochastic volatility models, one important instance being the GARCH models.

Figure (1) shows the autocorrelation and partial autocorrelation of the weekly (log) return in the DAX and its square, respectively. The dotted line shows the level of significance as given by the S-Plus function **acf**. All lags except the third show insignificant autocorrelation. Judging from both the autocorrelation function and the partial autocorrelation function, weekly DAX returns very much look like white noise. Looking at squared returns, however, significant autocorrelation up to for weeks can be seen. There also seems to be some autocorrelation of squared returns at 13 weeks (a quarter of a year).

In section 2, we review some facts about generalized hyperbolic distributions and normal variance-mean mixtures. We show in section 3 that certain GARCH-type models produce stationary marginal distributions that are generalized hyperbolic.

2 Generalized Hyperbolic Distributions as Normal Variance-Mean Mixture Distributions

Hyperbolic distributions were introduced by Barndorff-Nielsen (1977) in the context of the distribution of the size of sand particles taken from from the

Danish coast. Plots of the log histogram of their log size seemed to have linear tails and could be very well fitted by hyperbolae. This naturally led to the definition of the hyperbolic distribution, whose log density is a hyperbola:

$$\log h(x;\alpha,\beta,\mu,\delta) = c(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)). \tag{1}$$

(c is a norming constant, μ is a location parameter, δ a scale parameter, and $(\alpha+\beta)x$ and $(-\alpha+\beta)x$ are the asymptotes of the hyperbola.)

Empirical distributions that apparently have exponentially decreasing tails abound. The examples given in Barndorff-Nielsen (1977) include (log) diamond sizes from mining areas in South Africa, (log) personal income in Australia, and measurements of the velocity of light by Michelson.

There is a natural multivariate extension of (1) were the density is a hyperboloid. It turns out that the marginal and conditional distributions of multivariate hyperbolic distributions are not necessarily hyperbolic. Instead, hyperbolic distributions can be embedded into the class of *generalized hyperbolic distributions*, which is invariant under margining , conditioning, and affine transformations. Many important distributions are in this class or a limiting case of it. Namely, the Gaussian distribution, Student's t-distribution, the Laplace-distribution, the gamma distribution, and the reciprocal gamma distribution. (See Barndorff-Nielsen and Blaesild (1981)[p.20].)

Definition (Barndorff-Nielsen et al.; 1982)
A random variable $X \in \mathbf{R}^n$ is said to be distributed according to a normal variance-mean mixture with location μ, drift β, structure matrix Δ, and mixing distribution F, if there is a random variable u with a distribution F on $[0,\infty)$ and the conditional distribution of X under u is normal:

$$P^{X|u} = N_n(\mu+u\beta, u\Delta).$$

$\mu,\beta \in \mathbf{R}^n$, Δ is symmetric and positive definite, $\det(\Delta)=1$. We denote this distribution by $\mathrm{NVMM}(\mu,\beta,\Delta,F)$. (The condition $|\Delta|=1$ makes the choice of parameters unique and excludes the case $\Delta=0$, where any distribution F on $[0,\infty)$ could be written as $\mathrm{NVMM}(0,1,0,F)$. For $n=1$, the parameter $\Delta=1$ is irrelevant and we write $\mathrm{NVMM}(\mu,\beta,F)$.)

If ϕ is the characteristic function of F, the characteristic function ψ of $\mathrm{NVMM}(\mu,\beta,\Delta,F)$ is given by

$$\psi(t) = e^{i<t,\mu>}\phi(<\beta,t> + \frac{i}{2} < t, \Delta t >), \tag{2}$$

as is easily seen. A simple consequence of (2) is that the sum of i.i.d. NVMM variables with the same β and Δ is again a NVMM distribution:

$$NVMM(\mu,\beta,\Delta,F)^{*n} = NVMM(n\mu,\beta,\Delta,F^{*n}).$$

Furthermore, this class of mixtures is closed under margining, conditioning and affine transformations. Details are given in (Barndorff-Nielsen et al.; 1982, p.147).

Generalized hyperbolic distributions can now be defined as normal variance-mean mixtures were the mixing distribution is a *generalized inverse Gaussian distribution* $N^{-}(\lambda, \chi, \psi)$ that is defined by having the following density:

$$f(x) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\chi})} x^{\lambda-1} \exp\{-\frac{1}{2}(\chi x^{-1} + \psi x)\}. \qquad (x > 0)$$

(K_λ is a Bessel function. The parameter domain is $\lambda \in \mathbf{R}$, $\chi > 0$, $\psi > 0$. Additionally, $\chi = 0$ is allowed for $\lambda > 0$ and $\psi = 0$ is allowed for $\lambda < 0$.) Let

$$\begin{aligned} H(\lambda, \alpha, \beta, \mu, \delta, \Delta) &= NVMM(\mu, \beta\Delta, \Delta, N^{-}(\lambda, \delta^2, \kappa^2)), \\ & \kappa^2 = \alpha^2 - <\beta, \Delta\beta> \end{aligned}$$

denote the generalized hyperbolic distribution. The density and characteristic function of the generalized hyperbolic distribution are given in Barndorff-Nielsen et al. (1982)[p.148]. For $\lambda = (n+1)/2$, the density simplifies to

$$\frac{(\kappa/\delta)^\lambda}{(2\pi)^{\lambda-1} 2\alpha K_\lambda(\delta\kappa)} e^{-\alpha\sqrt{\delta^2 + <x-\mu, \Delta^{-1}(x-\mu)>} + <\beta, x-\mu>},$$

whose logarithm is a hyperboloid in x.

Maximum likelihood estimation for hyperbolic distributions with parameters $H(1, \alpha, \beta, \mu, \delta)$ is discussed in Barndorff-Nielsen and Blaesild (1981), where it is noted that the computation of the MLE is "from a theoretical as well as from a computational point of view rather unpleasant", [p.29]. Namely, the log-likelihood function may have saddle points.

Back to the feature that gave rise to the definition of the hyperbolic distribution. It is shown in Barndorff-Nielsen and Blaesild (1981)[pp.34] that for the density h of the univariate generalized hyperbolic distribution

$$h(x; \lambda, \alpha, \beta, \mu = 0, \delta) \sim |x|^{\lambda-1} e^{-(\pm\alpha-\beta)x} \qquad \text{as} \qquad x \to \pm\infty$$

for $\lambda > 0$. It is dubious whether one can see the effect of the power $|x|^{\lambda-1}$ in the log histogram. That is, generalized hyperbolic distributions very much "look like" hyperbolic distributions in the tails. That so much is known about generalized hyperbolic distributions makes it worthwhile investigating whether *normal variance-mean mixture distributions* implied by certain stochastic models are in fact *generalized hyperbolic*.

3 Normal Variance-Mean Mixture Distributions Generated by Stochastic Processes

If W is a one-dimensional Wiener-process and τ is an independent random time with distribution F, then

$$\mu + \beta\tau + W_\tau \sim \mathrm{NVMM}(\mu, \beta, F).$$

Moreover, if $\{\tau_a\}$ is an increasing stochastic process which is independent of W, then $X_a := \mu + \beta\tau_a + W_{\tau_a}$ is a stochastic process whose increments are mixed normal. For financial markets, $a \to \tau_a$ can be thought of as a random time change mapping calendar time to "internal market time" or "operational time". The generalized hyperbolic distribution with $\lambda = -\frac{1}{2}$ is now naturally obtained by letting τ_a be a certain first hitting time of an independent Wiener-process B:

$$\tau_a := \inf_{t\geq 0}\{\kappa t + B_t \geq a\delta\}.$$

τ_a is a Levy-process, i.e. it has independent increments. The increments are inverse Gaussian distributed: $\tau_b - \tau_a \sim N^-(-\frac{1}{2}, ((b-a)\delta)^2, \kappa^2)$. Now, $X_a = W_{\tau_a}$ is a Levy-process with generalized hyperbolic increments. X was termed *Gaussian-inverse Gaussian process* and its Levy decomposition was given by Barndorff-Nielsen (1994). X is a pure jump process, and the Levy measure is not integrable at 0. That means, X has infinitely many (small) jumps in any time interval. X can be thought of as a limit of compound Poisson processes.

Normal variance-mean mixtures also appear in stochastic volatility models were log returns are – conditionally on some stochastic volatility σ – multivariate Gaussian:

$$R_k = \mu + \sigma_k^2\beta + \sigma_k z_k.$$

σ is a stochastic process and z_k is Gaussian white noise (z_k i.i.d. $N_n(0, \Delta)$; $\beta, \mu \in \mathbf{R}^n$). If σ^2 is a stationary process with marginal distribution F and independent of (z_k), then R is a stationary process with marginal distribution $\mathrm{NVMM}(\mu, \beta, \Delta, F)$.

One simple example of a stochastic model for $r_t := \sigma_t^2$ is

$$dr_t = (\omega - \theta r_t)dt + c r_t dW_t, \tag{3}$$

where W_t is an independent Wiener-process. In the framework of Nelson (1990), (3) is a weak limit of the volatility process in the GARCH(1,1)-model

$$r_{(k+1)h} = \omega_h + r_{kh}(\beta_h + c_h B_{kh}^2)$$

for $h \to 0$, and $B_{kh} \sim$ i.i.d.$N(0, h)$, $\omega_h = \omega h$, $\beta_h = 1 - c\sqrt{h/2} - \theta h$, $c_h = c/\sqrt{2h}$. Nelson (1990) shows that the stationary distribution of (3) is reciprocal gamma if $1 + 2\theta/c^2 > 0$ and $\omega > 0$. That is, if $1/r_0 \sim \Gamma(1 + 2\theta/c^2, 2\omega/c^2)$, then r is (strictly) stationary.[2] Now, the stationary distribution of R is generalized hyperbolic $H(-(1 + 2\theta/c^2), \beta, \beta, \mu, 2\sqrt{\omega}/c)$. (Note that this one has tails $\sim |x|^{\lambda-1}$ for $x \to \infty$ and $\sim e^{2\beta x}$ for $x \to -\infty$ which is not quite what we observe.)

Yet another simple volatility model is

$$dr_t = (\omega - \theta r_t)dt + c\sqrt{r_t}dW_t.$$

Its stationary distribution is a gamma distribution: $\Gamma(2\omega/c^2, 2\theta/c^2)$. Hence, the stationary distribution of R is generalized hyperbolic with parameters $H(2\omega/c^2, \sqrt{4\theta/c^2 + \beta^2}, \beta, \mu, 0)$.

4 Conclusion

(i) For stock returns, Gaussian tails ($\sim e^{-x^2}$) are too light while Pareto-stable tails ($\sim |x|^{\alpha}$) seem to be too heavy. The tails of generalized hyperbolic distributions ($\sim |x|^{\alpha} e^{-\lambda x}$) – which Barndorff-Nielsen calls *semi-heavy* – seem to fit just right to the distribution of (log) stock returns.

(ii) Many stochastic volatility models produce – almost by definition – log returns whose stationary distribution is a *normal variance-mean mixture*. As the subclass of generalized hyperbolic distributions has several nice properties and is well-studied, it is worthwhile to investigate whether new (or old) stochastic volatility models produce log returns that are in fact generalized hyperbolic.

(iii) The quite numerous approaches that successfully fit stock returns with heavy-tailed distributions but assume (explicitly or implicitly) independent returns overlook that there is strong evidence that stock returns are not independent. Moreover, volatility can be predicted and these models ignore that opportunity.

References

Barndorff-Nielsen, O. (1977). Exponentially decreasing distributions for the logarithm of particle size, *Proc. R. Soc. London A* **353**: 401–419.

Barndorff-Nielsen, O. (1994). Gaussian-inverse Gaussian processes and the modelling of stock returns, presented at the 2. Workshop on Stochastics and Finance 1994 in Berlin.

[2] Let $\Gamma(\lambda, b)$ denote the gamma distribution with density $(b^{\lambda}/\Gamma(\lambda))x^{\lambda-1}e^{-bx}$, $x > 0$. Then $\Gamma(\lambda, b) = N^{-}(\lambda, 0, 2b)$ and, if $X \sim \Gamma(\lambda, b)$, then $X^{-1} \sim N^{-}(-\lambda, 2b, 0)$.

Barndorff-Nielsen, O. and Blaesild, P. (1981). Hyperbolic distributions and ramifications: Contributions to theory and application, *in* C. T. et al. (ed.), *Statistical Distributions in Scientific Work*, Vol. 4, D. Reidel Publishing Company, pp. 19–44.

Barndorff-Nielsen, O., Kent, J. and Sørensen, M. (1982). Normal variance-mean mixtures and z distributions, *International Statistical Review* **50**: 145–159.

Eberlein, E. and Keller, U. (1994). Hyperbolic distributions in finance, Research report, Institut für Mathematische Stochastik, Universität Freiburg. presented at the 2. Workshop on Stochastics and Finance 1994 in Berlin.

Fama, E. (1965). The behavior of stock market prices, *Journal of Business* **38**: 34–105.

Harrison, J. and Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Applications* **11**: 215–260.

Küchler, U., Neumann, K., Sørensen, M. and Streller, A. (1994). Stock returns and hyperbolic distributions, Discussion Paper 23, Sonderforschungsbereich 373, Humboldt-Universität zu Berlin. presented at the 2. Workshop on Stochastics and Finance 1994 in Berlin.

Mahieu, R. J. (1995). *Financial Market Volatility*, Universitaire Pers Maastricht.

Mandelbrot, B. and Taylor, H. (1967). On the distribution of stock price differences, *Operations Research* **15**: 1057+.

Mittnik, S. and Rachev, S. (1993). Modeling asset returns with alternative stable distributions, *Econometric Reviews* **12**(3): 261–330.

Nelson, D. B. (1990). ARCH models as diffusion approximations, *Journal of Econometrics* **45**: 7–38.