# Super-Hedging and Arbitrage Pricing in Markets with Transaction Costs and Trading Constraints

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## Abstract

The arbitrage pricing principle has been used to derive price relations like the Black-Scholes formula and Heath-Jarrow-Morton models in the context of frictionless markets and unconstrained trading. These relations may or may not be good approximations to reality. Even if, they are certainly not enforced by *real world* arbitrage because of transaction costs and trading constraints.

Recently, results on several aspects of the bounds implied by the absence of arbitrage in realistic models have been established by Dermody and Rockafellar (1991); El Karoui and Quenez (1992); Cvitanic and Karatzas (1993); Dermody and Rockafellar (1995); Jouini and Kallal (1995b,a), and Cvitanic and Karatzas (1996). Using these results and contributing some new ones, this paper attempts to put together a "realistic arbitrage pricing theory" in a discrete-time market like that in (Harrison and Pliska; 1981).

Under transaction costs and trading constraints, several aspects change fundamentally compared to the classical theory. The 1-to-1 relation between price functionals and martingale measures can be maintained only if there is a numeraire without transaction costs and trading constraints. A version of the fundamental theorem of finance (present value principle) can be maintained only of the set of admissible trading strategies is a cone and transaction costs are linear. Instead of looking for exact solutions to the present value equation (martingale measures) one should look for approximate solutions to the present value equation, with the metric depending on the structure of the transaction costs.



**Keywords:** arbitrage bounds, super-hedging, linear programming, duality theory, transaction costs, minimax problems
**JEL classification:** E43, C14, C61
**Mathematics Subject Classification:**



### Credits

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## 1 Introduction

Starting with the Black-Scholes Formula (Black and Scholes; 1973), the theory of *Mathematical Finance* has been dominated by Brownian-motion-driven models and other models that are inherently *complete* (Harrison and Kreps; 1979; Harrison and Pliska; 1981; Cox and Rubinstein; 1985; Heath et al.; 1992). This "complete markets theory" suggests two things to the options trader: that it is possible to duplicate (hedge) any derivative claim by clever trading in the underlying; and that the optimal hedging strategy is to continuously adapt one's stake in the underlying to the option's delta (delta-hedging). Academically speaking, markets are assumed to be complete under continuous trading. In this theory, prices of derivative assets are determined by arbitrage.

If one takes this theory literally, it is is both practically and theoretically misleading. In practice, most derivatives are not duplicable, especially if transaction costs are considered. The tremendous success of financial derivatives markets like the Chicago Board of Trade or the London International Financial Futures Exchange shows that derivatives are far from being redundant. Moreover, approximate delta-hedging may prove very costly compared to other hedging principles (Bensaid et al.; 1992). In theory, the very principle of arbitrage yields *inequalities* among asset prices. Given the prices of the underlying securities, this leads to *bounds* on the price of the derivative.

Well, practitioners don't take this theory literally. Modifications of the Black-Scholes-model are successfully used as a vehicle to define price functionals and hedging strategies for many types of option contracts. Versions or modifications of the models of Cox et al. (1985) and Heath et al. (1992) are used to value interest rate derivatives. It seems that price relations derived by fitting certain arbitrage-free, complete models to the real world are in fact good approximations. Now, we have the at least academically unsatisfying situation that the theory used to teach this "complete market model approach" does not explain why it works in practice. In other words, Black-Scholes works, but not because of the arbitrage pricing principle. A *statistical* option valuation theory that regards the problem of finding a "good" price functional as a problem of *estimation* or *filtering*, respectively, could explain why certain models fit better than others.

An alternative to looking for one single price of the derivative relative to the underlying is to consider the price bounds that are enforced by arbitrage. This is very appealing from both an economic and a probabilistic viewpoint: the assumption that markets are arbitrage-free is weak and holds in practice almost all the time. Arbitrage bounds neither depend on preferences nor on probabilities – which are never known for sure – but only on the set of all possible scenarios. Recently, results on several aspects of arbitrage bounds have been established by Dermody and Rockafellar (1991); El Karoui and Quenez (1992); Cvitanic and Karatzas (1993); Dermody and Rockafellar (1995); Jouini and Kallal (1995a,b), and Cvitanic and Karatzas (1996)), which builds a basis for a "realistic arbitrage pricing theory".

Previously, arbitrage bounds for stock options were found to be uninteresting from an academic and useless from a practical viewpoint, as the super-hedging strategy for a European call on a stock is to buy one share of the underlying (Davis and Clark; 1995). In fixed income markets, however, there are hundreds of "derivatives" on one underlying: the money market rate. Correspondingly, arbitrage bounds are quite tight for a range of securities (Jaschke; 1995).

Computation of arbitrage bounds is essentially worst case analysis and as such could be used by regulatory bodies as a tool to assess the riskiness of financial institution's portfolios.

This paper gives a compact review of the arbitrage theory of incomplete markets and develops new concepts in a discrete time setting. Section 2 introduces notation. Section 3 develops "classical" results in an incomplete market with dividend-paying securities. While equivalent martingale measures are the key concept in the classical theory, section 4 shows that they loose this role to *consistent price functionals* if transaction costs are introduced. Section 5 proves a generalization of the "Fundamental Theorem of Finance" to markets with transaction costs and trading constraints. Section 6 provides a characterization of the consistent price functionals appearing in the sections 4 and 5 as "good-fit functionals".

# 2 A Discrete Time Market

## 2.1 Notation

The following setup and notation are fairly close to (Harrison and Pliska; 1981), with the extension that securities pay dividends and trading generates transaction costs.

We consider a discrete, finite set of trading dates $[0, 1, \dots, T]$ and a finite set $\Omega$ of scenarios. We think of $\omega \in \Omega$ as a possible path ($\omega = (\omega_1, \dots, \omega_T)$). There is an (unknown) probability measure ($P(\{\omega\}) \geq 0$) and the natural filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \in \{0,\dots,T\}}$, $\mathcal{F}_t = \sigma(\mathrm{proj}_s, s \in 1, \dots, t)$, $\mathrm{proj}_t(\omega) = (\omega_1, \dots, \omega_t)$. We also have a filtration $\mathbf{G} = (\mathcal{G}_t)_t$, $\mathcal{G}_t \subset \mathcal{F}_t$ that represents the information known to agents. It will be convenient to assume $\mathcal{F}_0 = \mathcal{G}_0 = \{\emptyset, \Omega\}$. Stochastic cash flows are $\mathbf{G}$-adapted processes $Z$ with $Z_0 = 0$ and $\Delta Z_t = Z_t - Z_{t-1}$ is the amount paid at time $t$. (For any process $X$ we define $\Delta X := X_t - X_{t-1}$.) We have $m$ securities with *price* *processes* $S = (S_t^1, \dots, S_t^m)_{t\in\{0,\dots,T\}}$ and *cumulative dividend processes* $D = (D^1, \dots, D^m)_{t\in\{0,\dots,T\}}$[1] with the interpretation that $S_t$ is the ex-dividend price vector at time $t$ and constant on the time interval $[t, t+1)$. $\Delta D_t^i$ is the dividend (or coupon) paid by security $i$ at time $t$. $D_{-1} := 0$. $S$ and $D$ are $\mathbf{G}$-adapted. Note that $\mathbf{G}$ need not be the $\sigma$-algebra generated by $S$ and $D$. Investors may have information from processes that are not traded.

*Trading strategies* $\phi$ are $\mathbf{G}$-predictable processes with the interpretation that $\phi_t^i$ is the number of securities of type $i$ held over the interval $(t-1, t]$. At time $t$ investors

- still hold $\phi_t$,
- receive dividends $\phi_t \Delta D_t$ on their holdings,[2]
- observe the value $\phi_t S_t$ of their portfolio at time-$t$ ex-dividend prices,
- adjust their portfolio from $\phi_t$ to $\phi_{t+1}$ at a cost of $(\phi_{t+1} - \phi_t)S_t$.

Define $(\Delta^+\phi)_t := \phi_{t+1} - \phi_t$. We will also write $\phi_{t+} := \phi_{t+1}$ to remind ourselves that $\phi_{t+1}$ is chosen with the knowledge of $\mathcal{G}_t$ and just after time-$t$ prices and dividends are observed. $\phi_t$ is defined for $t \in \{0, \dots, T+1\}$ where $\phi_0 \in \mathbf{R}$ is the initial portfolio and $\phi_{T+1} \in L(\mathcal{G}_T)$[3] is the terminal time portfolio. The set of trading strategies $\Phi$ might be restricted. For example, $\phi^i \geq 0$ (no short-selling in security $i$) or $\sum_i |S^i \phi^i| \leq \text{const}$ (limits on volume). $\Phi$'s exact definition will differ from section to section. If nothing else is said, $\Phi$ is some subset of the set of $\mathbf{G}$-predictable processes that contains at least the zero trading strategy.

Define the *value process* $V_t(\phi) := \phi_{t+} S_t$, which is $\mathbf{G}$-adapted.[4] The *cash flow* $C(\phi)$ *generated by a trading strategy* $\phi$ is given by

$$\Delta C_t(\phi) = \phi_t \Delta D_t - (\Delta^+\phi_t) S_t - \Delta A_t(\phi), \tag{1}$$

$C_{-1} := 0$. Cumulative transaction costs $A_t(\phi)$ are assumed to be

$$\Delta A_t(\phi) = \mathrm{tac}_t(\Delta^+\phi_t) = \sum_{i=1}^{m} \mathrm{tac}_t^i(\Delta^+\phi_t^i), \tag{2}$$

$A_{-1} := 0$. We could consider

- *linear transaction costs*: $\mathrm{tac}_t^i(x) = a_t^i x^+ + b_t^i x^-$,
- linear plus fixed costs: $\mathrm{tac}_t^i(x) = a_t^i x^+ + b_t^i x^- + 1_{\{x^i \neq 0\}} c_t^i$, or

[1] In this paper $x^m$ denotes the $m$'th component of a vector. If we mean exponentiation, it will be clear from the context.

[2] For two vectors $x$ and $y$ let $xy$ denote the scalar product.

[3] Let $L(\mathcal{G}_t)$ denote the set of $\mathcal{G}_t$-measurable random variables.

[4] This differs from (Harrison and Pliska; 1981) for trading strategies that are not self-financing.

- realistic transaction costs: $\text{tac}_t^i(.)$ is discontinuous at 0 (because of fixed costs), has concave shape for small transactions (because of decreasing rates for higher volumes), and convex shape for big transactions (because of additional market impact costs for very big transactions).

Here, $\text{tac}_t^i$, $a_t^i$, $b_t^i$, and $c_t^i$ are $\mathbf{G}$-adapted processes.

We have the following difference equation for $V$:

$$\Delta V_t(\phi) = \phi_t \Delta(S+D)_t - \Delta C_t(\phi) - \Delta A_t(\phi). \tag{3}$$

We call $\phi$ *self-financing* if $C(\phi) \equiv 0$ P-a.s.

We do not generally assume $S \geq 0$. (Think of the value of an interest rate swap contract, for example.) We say that security $i$ can act as a *numeraire* if $S^i > 0$ and $D \equiv 0$ P-a.s. In this case we define the corresponding *discounting process* $\beta_t := S_0^i / S_t^i$.

## 2.2 Super-Hedging and Arbitrage Pricing

Define $Z \succeq 0$ if $Z$ is increasing, which defines an ordering on the set of cash flows. Correspondingly, we define a *nonnegative price functional* $\pi \succeq 0$ as a functional that is nonnegative on all increasing processes.

We call a trading strategy $\phi \in \Phi$ a *super-hedging strategy* for a cash flow $Z$ if the process $C(\phi) - Z$ is increasing and $\phi_{T+} = 0$ P-a.s. Let $\Phi^+(Z)$ denote the set of these strategies.

We define the *arbitrage upper bound* for the price of a cash flow $Z$ relative to the market $(\Omega, \mathbf{G}, P, D, S)$, the set of trading strategies $\Phi$, and transaction costs tac(.) as

$$\overline{\pi}(Z) = \inf_{\phi \in \Phi^+(Z)} \{-C_0(\phi) \mid \phi_0 = 0\}. \tag{4}$$

We call a solution $\phi^*$ of (4) a *least cost super-hedging strategy* for $Z$.

$Z$'s *arbitrage lower bound* is

$$\underline{\pi}(Z) = \sup_{\phi \in \Phi^+(-Z)} \{C_0(\phi) \mid \phi_0 = 0\}. \tag{5}$$

Obviously, $\underline{\pi}(Z) = -\overline{\pi}(-Z)$ holds.

On a finite probability space $\Omega$ any cash flow is bounded. So, if there is a numeraire, any cash flow can be super-hedged. More formally:

**Lemma 1** *If a numeraire exists (security $i$) and $\Phi$ contains the cone*

$$\{\phi \textit{ predictable } \mid \phi^i \geq 0, \phi^k = 0 \, \forall k \neq i\}$$

*then $\Phi^+(Z)$ is nonempty for all cash flows $Z$.*

*Remark* Unlike Dermody and Rockafellar (1991) we do not model the existence of fiat money in the definition of "$Y$ is at least as good as $Z$" ( = "$Y - Z$ is increasing"). To model the existence of cash, include an asset $S^1 \equiv 1, D^1 \equiv 0$ and impose the restriction $\phi^1 \geq 0$. To model the existence of an account with different borrowing and lending rates $r$ and $R$, include

$$\begin{aligned} S_t^1 &= S_0^1 \prod_{s=1}^{t}(1+r_s),\ D^1 \equiv 0, \\ S_t^2 &= S_0^2 \prod_{s=1}^{t}(1+R_s),\ D^2 \equiv 0, \end{aligned}$$

and impose $\phi^1 \geq 0$, $\phi^2 \leq 0$.

A market is called *strongly arbitrage-free* if there is no trading strategy $\phi$ with $\phi_0 = 0$, $\Delta C_t(\phi) \geq 0$ P-a.s. for all $t \in \{0, \ldots, T\}$ and $P(\Delta C_t(\phi) > 0) > 0$ for some $t$. Such a strategy would promise "maybe some profit in the future" at a zero cost. The market is called *(weakly) arbitrage-free* if $\overline{\pi}(0) = 0$. If $\overline{\pi}(0) < 0$, the corresponding trading strategy represents a "sure arbitrage profit now". Such strategies are called *arbitrage opportunities of the first type* and *second type*, in this order (Ingersoll; 1987, p.53). The notion of arbitrage in (Harrison and Pliska; 1981) corresponds to the first type. As we consider the second type only[5] we say that *NA holds* if the market is (weakly) arbitrage-free.

### Justification of the Name

If someone wants to buy $Z$ for a price $\pi(Z) > \overline{\pi}(Z)$ then

- sell Z at the price $\pi(Z)$, taking the obligation to pay $Z$ in the future,
- from $\pi(Z)$ set up a least cost super-hedging strategy $\phi$,
- pocket the difference $\pi(Z) - \overline{\pi}(Z)$, which is an arbitrage profit, since we start with nothing ($\phi_0 = 0$) and $C(\phi)$ covers the obligation $Z$ in any scenario.

An analogous argument holds for the arbitrage lower bound. If someone wants to sell $Z$ for a price $\pi(Z) < \underline{\pi}(Z)$

- buy $Z$, so we will receive $Z$ in the future,

[5]Arbitrage of the second type corresponds to optimization over closed sets while that of the first type does not. So it is more convenient to consider the second type. Also, arbitrage opportunities of the first type that or not arbitrage opportunities of the second type do virtually "not" occur in practice.

- set up a trading strategy $\phi$ whose cash flow is covered by $Z$ ($\Delta(Z + C(\phi)) \geq 0$),
- pocket the difference $\underline{\pi}(Z) - \pi(Z)$.

El Karoui and Quenez (1992) consider super-hedging of contingent claims (payoffs at the terminal time $T$ only) and call the implied pricing bounds *buying price* and *selling price*, respectively. Bensaid et al. (1992) name these bounds *manufacturing cost* and *revenue*, respectively. Dermody and Rockafellar (1991, 1995) consider deterministic cash flows and call the corresponding bounds *imputed long* and *short price*, respectively. Cvitanic and Karatzas (1993, 1996) consider the upper bound and call it *hedging price*. Jouini and Kallal (1995b,a) use the name arbitrage bounds.

### 2.3 Some Properties of the Arbitrage Bounds

In this subsection we state some rather obvious facts about arbitrage bounds. Call $f$ mapping a vector space $M$ to another vector space *linear in scale* if $f(\alpha x) = \alpha f(x)$ for all $x \in M$ and $\alpha > 0$.

**Lemma 2** *If there are linear transaction costs and $\Phi$ is a cone then $\overline{\pi}$ and $\underline{\pi}$ are linear in scale.*

In a realistic model $\Phi$ might not be a cone, for example, if short-selling is allowed but restricted in size.

**Lemma 3** *If $\overline{\pi}$ is linear in scale then the maximal arbitrage profit $-\overline{\pi}(0)$ is either 0 or $\infty$.*

**Lemma 4** *If the functions $x \mapsto tac_t^i(x)$ are convex and the set $\Phi$ is convex then $\overline{\pi}$ is convex and $\underline{\pi}$ is concave.*

### 2.4 Arbitrage Bound Processes

Let $\Phi_t^+(Z)$ denote the set of trading strategies that super-hedge $Z$ on the interval $[t+1, T]$:

$$\Phi_t^+(Z) = \{\phi \in \Phi \,|\, \Delta(C(\phi) - Z)_s \geq 0 \,\forall s \in \{t+1, \ldots, T\}, \phi_{T+} = 0\}.$$

Define

$$\overline{\pi}_t(Z) := \inf_{\phi \in \Phi_t^+(Z)} \{V_t(\phi) + \Delta A_t(\phi) \,|\, \phi_t = 0\} \tag{6}$$

and $\underline{\pi}_t(Z) := -\overline{\pi}_t(-Z)$. The upper and lower arbitrage bound processes are $\mathbf{G}$-adapted. (Observe that $-\Delta C_t(\phi) = V_t(\phi) + \Delta A_t(\phi)$ under $\phi_t = 0$.)

In the presence of transaction costs or short-selling constraints it is necessary to consider the initial portfolio in the optimization problem (see also (Dermody and Rockafellar; 1995)). Define

$$\overline{\pi}_t(Z,x) = \inf_{\phi \in \Phi_t^+(Z)} \{V_t(\phi) + \Delta A_t(\phi) \mid \phi_t = x\} \tag{7}$$

as the *customized risk-free upper bound* valid for an investor with the portfolio $x$ at time $t$.

$\underline{\pi}_t(Z,x) = -\overline{\pi}_t(-Z,x)$

is the *customized risk-free lower bound* valid for an investor with the portfolio $x$ at time $t$. ($\overline{\pi}_t(Z)$ values the remaining cash flow $(Z_{t+1},\ldots,Z_T)$, which corresponds to our convention that prices are ex-dividend.) $-\overline{\pi}_0(0,x) = \underline{\pi}_0(0,x)$ is the *maximal risk-free portfolio improvement profit* possible with an initial portfolio $x$.

*Remark* From a mathematical perspective one could call $\underline{\pi}(0,x)$ an arbitrage profit for all $x$ since this is a completely risk-free profit. In economics, however, an arbitrage has a connotation of being possible without initial wealth. If there are no transaction costs and no trading constraints, this distinction is not necessary and $\overline{\pi}_t(Z,x) = \overline{\pi}_t(Z,0)$ for all $x$. Dermody and Rockafellar (1995) call the analogue of $\overline{\pi}(.,x)$ and $\underline{\pi}(.,x)$ *customized long* and *short prices.*

# 3 The Ideal Market

In this section we develop a dynamic arbitrage pricing theory in discrete time. Ideas are owed to (Harrison and Kreps; 1979; Harrison and Pliska; 1981; Dermody and Rockafellar; 1991; El Karoui and Quenez; 1992; Cvitanic and Karatzas; 1993). The main point will be to show

$$\overline{\pi}_t(Z) = \sup_{Q\in\mathcal{P}} \mathrm{E}_Q[\sum_{s=t+1}^{T} \frac{\beta_s}{\beta_t}\Delta Z_s \mid \mathcal{G}_t], \tag{8}$$

$$\underline{\pi}_t(Z) = \inf_{Q\in\mathcal{P}} \mathrm{E}_Q[\sum_{s=t+1}^{T} \frac{\beta_s}{\beta_t}\Delta Z_s \mid \mathcal{G}_t], \tag{9}$$

which can be seen as a present value principle for a possibly incomplete but otherwise ideal market.

*Ideal market* means:

- There are no transaction costs,

- there are no restrictions on trading strategies (that is, $\Phi$ is the set of all $\mathbf{G}$-predictable processes with $\phi_{T+} = 0$), and
- all investors observe the same set of information ($\mathbf{G}$) and receive the same cash flows from a given strategy (i.e., there are no taxes).

We also assume the existence of a numeraire $S^1 > 0$ and define $\beta_t = S_0^1 / S_t^1$.

$\Phi$ is a vector space. Let $\mathbf{D}$ denote the set of all $\mathbf{G}$-adapted processes, where $P$-indistinguishable processes are identified. $C : \Phi \to \mathbf{D}$ is a linear mapping.

A functional $\pi : \mathbf{D} \to \mathbf{R}$ is called a *consistent price system* if

- $\pi$ is linear,
- $\pi$ is nonnegative ($\pi \succeq 0$), and
- $\pi(C(\phi)) = V_0(\phi)$ for all $\phi \in \Phi$.

Let $\Pi$ denote the set of consistent price systems.

A probability measure $Q$ that is defined on $(\Omega, \mathcal{G}_T)$ and is absolutely continuous with respect to $P$ ($Q \ll P|_{\mathcal{G}_T}$) is called a *martingale measure* if the *discounted gain process*

$$\tilde{S} = \beta S + \sum \beta \Delta D \quad ^{6}$$

is a $(\mathbf{G}, Q)$-martingale:

$$\mathrm{E}_Q[\beta_{t+1}(S_{t+1} + \Delta D_{t+1}) \,|\, \mathcal{G}_t] = \beta_t S_t. \tag{10}$$

Let $\mathcal{P}$ denote the set of martingale measures.

**Lemma 5** *For all $Q \in \mathcal{P}$ and all $\phi \in \Phi$*

$$V_0(\phi) + \sum \phi \Delta \tilde{S} = \beta V(\phi) + \sum \beta \Delta C(\phi) \tag{11}$$

*is a $Q$-martingale.*

Proof. Check equation (11) by taking differences. The left hand side is a martingale because the stochastic sum (integral) $\sum \phi \Delta X$ is a martingale for any martingale X and any predictable integrand $\phi$. $\square$

**Theorem 6** *There is a one-to-one relation between consistent price systems and martingale measures on $(\Omega, \mathcal{G}_T)$ by*

$$\begin{aligned} \pi(Z) &= E_Q[\sum^{T} \beta \Delta Z] \quad \textit{and} && (12) \\ Q(A) &= \pi((0, \ldots, 0, 1_A / \beta_T)). && (13) \end{aligned}$$

[6] We will frequently drop the summation index. Then $\sum \zeta \Delta X$ denotes the process $(\sum_{s=1}^{t} \zeta_s \Delta X_s)_{t \in \{0,1,\ldots,T\}}$ whose value is 0 at time 0.

Proof. $\pi \in \Pi$ given.
That $\pi$ is nonnegative and linear ensures that $Q$ defined by (13) is a measure. To see that $Q \ll P$, recall that $\pi$ is in fact a functional on the equivalence classes induced by $P$-indistinguishability. This ensures that $Q(A)$ defined by (13) is zero for all $\pi \in \Pi$ and $P$-null sets $A$. Define $\phi_t^k = 1_{\{k=1\}} 1_{\{t \in (0,T]\}} 1_{\{\omega \in \Omega\}}$; i.e., hold the numeraire throughout $[0, T]$. Apply the consistency equation to $\phi$ and observe $Q(\Omega) = 1$, that is $Q$ is a probability measure. Now define the trading strategy that starts with nothing, buys $X$ shares of security $k \neq 1$ at time $t$ and sells these shares at time $t+1$ ($\phi_s^j = X 1_{\{j=k\}} 1_{\{s=t+1\}}$). It generates the cash flow:

| time | cash flow $\Delta C(\phi)$ |
|---|---|
| $t$ | $-XS_t^k$ |
| $t+1$ | $X(S_{t+1}^k + \Delta D_{t+1}^k)$ |

This trading strategy is $\mathbf{G}$-predictable for any $\mathcal{G}_t$-measurable random variable $X$. We have

$$\sum^T \beta C(\phi) = X(\beta_{t+1}(S_{t+1}^k + \Delta D_{t+1}^k) - \beta_t S_t^k) = X \Delta \tilde{S}_{t+1}^k.$$

Since $\pi$ is consistent, $\mathrm{E}_Q[X \Delta \tilde{S}_{t+1}^k] = 0 \quad \forall X \in L(\Omega, \mathcal{G}_t)$. Hence, $\tilde{S}^k$ is a $(\mathbf{G}, Q)$-martingale for all $k \neq 1$. $\tilde{S}^1$ is a martingale anyway, as it is constant.
$Q \in \mathcal{P}$ given.
Obviously, (12) defines a nonnegative linear functional on $\mathbf{D}$. Equation (11), lemma 5 and $V_T(\phi) = 0$ imply that

$$\mathrm{E}_Q[\sum^T \beta \Delta C(\phi)] = V_0(\phi)$$

for all $\phi \in \Phi$ and $Q \in \mathcal{P}$. This ensures that $\pi$ defined by (12) is consistent.

It remains to show that (12) and (13) are inverse mappings. Start with a $Q \in \mathcal{P}$ and define $\pi(Z) = \mathrm{E}_Q[\sum^T \beta \Delta Z]$. Apply (13) to get

$$\tilde{Q} = \pi((0, \ldots, 0, \mathbf{1}_A / \beta_T))$$

and observe that $\tilde{Q}(A) = Q(A)$ for all $A \in \mathcal{G}_T$. On the other hand, start with a $\pi \in \Pi$ and define $Q$ by (13). $Q \in \mathcal{P}$ as shown above. Apply (12) to get

$$\tilde{\pi}(Z) = \pi((0, \ldots, 0, \frac{1}{\beta_T} \sum \beta \Delta Z)). \tag{14}$$

So we have to show that

$$\pi(Z) = \pi((0, \ldots, 0, \frac{1}{\beta_T} \sum \beta \Delta Z)) \tag{15}$$

holds for any consistent price system.

The difference of the cash flow $Z$ and the cash flow in the right hand side of (15) can be generated by a trading strategy $\phi \in \Phi$ that trades only in the numeraire ($\phi^k = 0, \forall k \neq 1$) and has a zero initial cost ($V_0(\phi) = 0$). Applying the condition $\pi(C(.)) = V_0(.)$ to $\phi$ shows that (15) holds for any $\pi \in \Pi$. This completes the proof. (Note that (13) and (12) are not generally inverse mappings but only for consistent price systems.) □

**Theorem 7** PRESENT VALUE PRINCIPLE
*The market is arbitrage-free if and only if a consistent price system exists. In this case the arbitrage bounds of the price of a cash flow $Z$ can be computed as its maximal and minimal present value, respectively, taken over all consistent price systems:*

$$\overline{\pi}_0(Z) = \max_{\pi \in \Pi} \pi(Z) = \max_{Q \in \mathcal{P}} E_Q[\sum^T \beta \Delta Z], \tag{16}$$

$$\underline{\pi}_0(Z) = \min_{\pi \in \Pi} \pi(Z) = \min_{Q \in \mathcal{P}} E_Q[\sum^T \beta \Delta Z]. \tag{17}$$

*The optimal values are attained. For the problem (16) this means that a trading strategy $\phi^* \in \Phi^+(Z)$ and a martingale measure $Q^* \in \mathcal{P}$ exist such that*

$$-C_0(\phi^*) = E_{Q^*}[\sum \beta \Delta Z]$$

*holds.*

Proof.

$$\overline{\pi}_0(Z) = \inf_{\phi \in \Phi} \{-C_0(\phi) \,|\, \Delta(C(\phi) - Z)_t \geq 0 \quad \forall t \geq 1 \,\text{P-a.s.}\}. \tag{18}$$

Writing the Lagrangian:

$$= \inf_{\phi \in \Phi} \sup_{\pi \in \mathbf{D}^*, \pi \succeq 0} \{-C_0(\phi) - \pi(C(\phi) - Z)\}. \tag{19}$$

where $\pi$ is a nonnegative linear functional on $D$. Now exchange inf and sup:

$$\geq \sup_{\pi \in \mathbf{D}^*, \pi \succeq 0} \left\{ \pi(Z) + \underbrace{\inf_{\phi \in \Phi} \{-C_0(\phi) - \pi(C)\}}_{(*)} \right\}. \tag{20}$$

$(*)$ is zero if and only if $\pi \in \Pi$, otherwise it is $-\infty$ (since $\Phi$ is a vector space and $C$ is linear).

$$= \max_{\pi \in \Pi} \pi(Z) \tag{21}$$

and by theorem (6)

$$= \max_{Q\in\mathcal{P}} \mathrm{E}_Q[\sum^T \beta\Delta Z]. \tag{22}$$

Since $\Omega$ is finite, $\Phi$ is a finite-dimensional vector space and the (strong) duality theorem of linear programming applies:

1. The primal (18) has a finite solution if and only if the feasible set in the dual (22) is nonempty.

2. In this case, a saddle point $(\phi^*, \pi^*)$ of the Lagrangian exists:

$$-C_0(\phi^*) = V_0(\phi^*) = \pi^*(Z) = \mathrm{E}_{Q^*}[\sum \beta\Delta Z].$$

The equation for $\underline{\pi}$ follows from $\underline{\pi}(Z) = -\overline{\pi}(-Z)$. □

**Theorem 8** *For any stochastic cash flow $Z$*

$$\overline{\pi}_t(Z) = \sup_{Q\in\mathcal{P}} E_Q[\sum_{s=t+1}^{T} \frac{\beta_s}{\beta_t}\Delta Z_s \,|\, \mathcal{G}_t] \quad \forall t P\text{-a.s.} \tag{23}$$

*holds and there exist $\phi^* \in \Phi^+(Z)$ and $Q^* \in \mathcal{P}$ such that*

$$V_t(\phi^*) = \overline{\pi}_t(Z) = \frac{1}{\beta_t} E_{Q^*}[\sum^T \beta\Delta Z \,|\, \mathcal{G}_t] \quad \forall t P\text{-a.s.} \tag{24}$$

*The analogous result holds for the arbitrage lower bound process $\underline{\pi}_t(Z)$.*

*Remark* Principally, there could be different optimal $\phi^* \in \Phi^+(Z)$ for every pair $(t,\omega)$ (if we know that the inf in (6) is attained $(t,\omega)$-wise). So this theorem states 3 facts:

1. the duality gap is zero;

2. inf and sup are attained;

3. there exists a pair $(\phi^*, Q^*)$ that is simultaneously optimal for all $(t,\omega)$.

Proof. We define the process

$$N_t(\phi) = \beta_t V_t(\phi) + \sum^t \beta\Delta Z,$$

which is a $Q$-supermartingale for all $Q \in \mathcal{P}$ and $\phi \in \Phi^+(Z)$. ($N$ is the martingale in lemma 5 plus a decreasing process.)
From $N_t \geq \mathrm{E}_Q[N_T|\mathcal{G}_t]$ and $V_T(\phi) = 0$ follows

$$\beta_t V_t(\phi) \geq \mathrm{E}_Q[\sum_{t+1}^T \beta\Delta Z|\mathcal{G}_t] \quad \forall t\text{P-a.s.}$$

Consider the saddle point $(\phi^*, Q^*)$ of the previous theorem

$$V_0(\phi^*) = \mathrm{E}_{Q^*}[\sum^T \beta\Delta Z],$$

or, equivalently, $N_0(\phi^*) = \mathrm{E}_{Q^*}[N_T(\phi^*)]$. A technical lemma[7] implies that $N(\phi^*)$ is a $Q^*$-martingale. Hence

$$\beta_t V_t(\phi^*) = \mathrm{E}_{Q^*}[\sum_{t+1}^T \beta\Delta Z|\mathcal{G}_t] \quad \forall t \mathrm{P}\text{-a.s.}$$

□

The assumption that $\Omega$ is finite allows to bypass many technical difficulties (as dealt with by El Karoui and Quenez (1992) and Cvitanic and Karatzas (1993)) and develop the main ideas of super-hedging and arbitrage pricing in a surprisingly simple and elegant way. The same approach will allow us to go far beyond (El Karoui and Quenez; 1992) and (Cvitanic and Karatzas; 1993) in terms of more realistic model assumptions in the next section.

# 4 Martingale Measures, Supermartingale Measures and Consistent Price Systems

Given a probability measure $Q \ll P$ and a nonnegative $\mathbf{G}$-adapted process $\beta$,

$$\pi(Z) = \mathrm{E}_Q[\sum^T \beta\Delta Z] \tag{25}$$

defines a nonnegative linear price functional on the set of cash flows $\mathbf{D}$. In fact, any nonnegative linear functional on $\mathbf{D}$ can be represented that way. We call a linear functional $\pi : \mathbf{D} \to \mathbf{R}$ a *consistent price system* if $\pi$ is nonnegative and

$$C_0(\phi) + \pi(C(\phi)) \leq 0 \quad \text{for all} \quad \phi \in \Phi. \tag{26}$$

Since the zero trading strategy is in $\Phi$, (26) is equivalent to

$$\max_{\phi\in\Phi}\{C_0(\phi) + \pi(C(\phi))\} = 0. \tag{27}$$

Let $\Pi$ denote the set of consistent price systems. The pair $(\beta, Q)$ is called consistent if it defines a consistent price system (by (25)).

[7]If $(X_t)_{t\in[0,T]}$ is a supermartingale and $E[X_T] = X_0$ then $X$ is a martingale.

**Theorem 9** *Under the following conditions there is a 1-to-1 correspondence between consistent price systems and absolutely continuous probability measures $Q$ on $(\Omega, \mathcal{G}_T)$ such that $(\frac{1}{S^1}, Q)$ is consistent:*

- *there is a numeraire ($D^1 \equiv 0, S^1 > 0$);*
- *trading in this security is unrestricted;*
- *transaction costs in this security are zero; and*
- *in the other securities trading is not restricted with respect to information. That means $\Phi$ contains at least the strategies that start with nothing, buy 1 share of security $k > 1$ at time $t$ in the event $A$ and sell these shares at time $t+1$ ($\phi_s^j(\omega) = 1_A(\omega) 1_{\{j=k\}} 1_{\{s=t+1\}}$), for all $k \neq 1$, $t \geq 1$, and $A \in \mathcal{G}_t$.*

Proof. This theorem is a direct generalization of theorem 6 and the proof proceeds in a similar way. The fourth condition says that the trading strategies needed in that proof are contained in $\Phi$.
(12) and (13) define inverse mappings: The condition that trading in the numeraire is unrestricted is sufficient to ensure the existence of a trading strategy $\phi$ that has zero initial cost $V_0(\phi) = 0$, generates the cash flow $C(\phi) = (0, -\Delta Z_1, \dots, -\Delta Z_{T-1}, \sum \beta \Delta Z)$ for arbitrary $Z$, and trades only in the numeraire ($\phi^k = 0 \, \forall k \neq 1$). That there are no transaction costs when trading in the numeraire ensures that $C(\phi) = -C(-\phi)$. Apply the consistency conditions to both $\phi$ and $-\phi$ and observe that (15) holds for all $Z \in \mathbf{D}$ and $\pi \in \Pi$.

Take a $\pi \in \Pi$ and define $Q$ by (13). $Q$ is a measure and $Q \ll P$. To see that $Q$ is a probability measure ($Q(\Omega) = 1$), apply the consistency condition (26) to the following two trading strategies:
$\psi$="put \$1 into the numeraire at time 0, sell at time $T$";
$\xi$="borrow \$1 from the numeraire at time 0, pay back at time $T$".
$(\beta, Q)$ is consistent because (13) and (12) define inverse mappings.

Take such a $Q$ that $(\beta, Q)$ is consistent. $\pi$ defined by (25) is consistent by definition. □

*Remark.* The link between consistent price systems and probability measures on $(\Omega, \mathcal{G}_T)$ breaks if there are different borrowing and lending rates or there are transaction costs in the numeraire. Then pairs $(\beta, Q)$ are needed to generate all consistent price functionals (Jouini and Kallal; 1995b).

In the rest of this section we assume that the conditions of theorem 9 are met. Then it is justified to define the set $\mathcal{Q}$ of *consistent probability measures* as those measures that correspond to a consistent price system.

**Lemma 10** *For all $Q \in \mathcal{P}$ and all* **G***-predictable processes $\phi$*

$$V_0(\phi) + \sum \phi \Delta \tilde{S} = \beta V(\phi) + \sum \beta \Delta C(\phi) + \sum \beta \Delta A(\phi) \tag{28}$$

*is a $Q$-martingale.*

Proof. Check equation (28) by taking differences. By definition of $\mathcal{P}$, $\tilde{S}$ is a $Q$-martingale for all $Q \in \mathcal{P}$. $\square$

**Lemma 11** *Every martingale measure is consistent ($\mathcal{P} \subset \mathcal{Q}$).*

Proof. Since $\sum \beta \Delta A(\phi)$ is an increasing process, lemma 10 implies that $\beta V(\phi) + \sum \beta \Delta C(\phi)$ is a $Q$-supermartingale for all $Q \in \mathcal{P}$. Since $V_T(\phi) = 0$,

$$\mathrm{E}_Q \left[ \sum \beta \Delta C(\phi) \right] \leq -C_0(\phi) \quad \forall \phi \in \Phi \, \forall Q \in \mathcal{P}.$$

$\square$

We call $Q \ll P$ a *supermartingale measure* if the discounted gain process $\tilde{S}$ is a $Q$-supermartingale. (Denote this set by $\mathcal{P}^{>}$.)

**Lemma 12** *If short-selling is forbidden in all securities except the numeraire, then every supermartingale measure is consistent. In this case we have $\mathcal{P} \subset \mathcal{P}^{>} \subset \mathcal{Q}$.*

Proof. Since $\sum \beta \Delta A(\phi)$ is an increasing process, $\phi^k \geq 0 \; \forall k \neq 1$, and $\tilde{S}^1 \equiv 1$, lemma 10 implies that $\beta V(\phi) + \sum \beta \Delta C(\phi)$ is a $Q$-supermartingale for all $Q \in \mathcal{P}^{>}$. Since $V_T(\phi) = 0$,

$$\mathrm{E}_Q \left[ \sum \beta \Delta C(\phi) \right] \leq -C_0(\phi) \quad \forall \phi \in \Phi \, \forall Q \in \mathcal{P}.$$

$\mathcal{P} \subset \mathcal{P}^{>}$ is obvious. $\square$

**Lemma 13** *If there are no transaction costs and $\Phi$ contains at least the cone of nonnegative predictable processes, then every consistent probability measure is a supermartingale measure. ($\mathcal{Q} \subset P^{<}$.)*

Proof. As in the proof of theorem 6, consider trading strategies $\phi$ that buy $X$ units ($X \in \mathcal{G}_t, X \geq 0$) of security $k$ at time $t$ and sell it at time $t+1$. For this $\phi$ we have $\sum^T \beta C(\phi) = X \Delta \tilde{S}^k_{t+1}$. The consistency condition (26) implies $\mathrm{E}_Q[X \Delta \tilde{S}^k_{t+1}] \leq 0$ for all $k$, $t$, $X \in \mathcal{G}_t$. So $\tilde{S}$ is a $Q$-supermartingale for all $Q \in \mathcal{Q}$. $\square$

The point is that, under transaction costs and trading restrictions, $\mathcal{Q}$ is the appropriate set to consider as opposed to $\mathcal{P}$. Empirically, $\mathcal{P}$ is often empty while $\mathcal{Q}$ is not (Jaschke et al.; 1996). Theoretically, $\mathcal{Q}$ is exactly the set we need for the present value principle.

# 5 The Present Value Principle

In this section we assume that transaction costs are linear and the set of admissible trading strategies $\Phi$ is a polyhedral cone. Consequently, $R : \Phi \to \mathbf{D}$ is linear, $C : \Phi \to \mathbf{D}$ is convex, $\overline{\pi}$ is convex and linear in scale.

**Theorem 14** PRESENT VALUE PRINCIPLE
*The market is arbitrage-free if and only if a consistent price system exists. In this case the arbitrage bounds of the price of a cash flow $Z$ can be computed as its maximal and minimal present value, respectively, taken over all consistent price systems:*

$$\overline{\pi}_0(Z) = \max_{\pi\in\Pi} \pi(Z) = \max_{(\beta,Q)\,consistent} E_Q[\sum^T \beta\Delta Z],$$

$$\underline{\pi}_0(Z) = \min_{\pi\in\Pi} \pi(Z) = \min_{(\beta,Q)\,consistent} E_Q[\sum^T \beta\Delta Z].$$

*The optimal values are attained.*

Both theorem and proof are almost identical to the proof of theorem 7:
Proof.

$$\overline{\pi}_0(Z) \quad = \quad \inf_{\phi\in\Phi}\{-C_0(\phi) \mid \Delta(C(\phi)-Z)_t \geq 0 \quad \forall t\geq 1\,\text{P-a.s.}\}. \tag{29}$$

Write the Lagrangian

$$= \quad \inf_{\phi\in\Phi} \sup_{\pi\in\mathbf{D}^*,\pi\succeq 0} \{-C_0(\phi) - \pi(C(\phi)-Z)\}. \tag{30}$$

Now exchange inf and sup:

$$\geq \quad \sup_{\pi\in\mathbf{D}^*,\pi\succeq 0} \left\{ \pi(Z) + \underbrace{\inf_{\phi\in\Phi}\{-C_0(\phi) - \pi(C(\phi))\}}_{(*)} \right\} \tag{31}$$

$(*)$ is zero if and only if $\pi \in \Pi$, otherwise it is $-\infty$ (since $\Phi$ is a cone and $C$ is linear in scale), so we get

$$= \quad \sup_{\pi\in\Pi} \pi(Z).$$

To see that $\overline{\pi}_0(Z)$ can be written as a linear programming problem introduce new processes $\Delta X_t = (\phi_{t+1}-\phi_t)^+$ and $\Delta x_t = (\phi_{t+1}-\phi_t)^-$. Now

$$\begin{aligned}\overline{\pi}_0(Z) \quad &= \inf\nolimits_{X-x\in\Phi} \quad \{-C_0(X,x) \mid \Delta(C(X,x)-Z)_t(\omega) \geq 0, \\ &\qquad\qquad X\geq 0, x\geq 0,\ \forall t\,\text{P-a.s.}\}\end{aligned}$$

where $C$ is linear in $(X, x)$. Now the arguments in the proof of theorem 7 apply:

1. The primal has a finite solution if and only if the feasible set in the dual is nonempty.

2. In this case, a saddle point $(\phi^*, \pi^*)$ of the Lagrangian exists:

$$-C_0(\phi^*) = \pi^*(Z)$$

□

It is seen from the proof that generally the existence of a consistent price system implies NA. If the duality gap is zero, the converse also holds. Generalizations of this result to nonlinear convex transaction costs are, however, not possible. They substantially change the form of the dual problem and the link between arbitrage bounds and consistent price systems breaks. Likewise do such restrictions on trading strategies that $\Phi$ is not a cone change the form of the dual problem.

# 6 Characterization of Consistent Price Systems as Good-Fit Functionals

In this section we assume that the conditions of theorem 9 are met, trading strategies are unrestricted ($\Phi = \{\phi \,|$ predictable and $\phi_{T+} = 0)$, transaction costs are linear and there are no transaction costs at the terminal time ($b_T \equiv 0$). Define

$$||X||_\infty = \sup_{\omega: P(\{\omega\})>0} \sup_{k,t} |X_t^k(\omega)|$$

and the *theoretical prices* induced by $Q$

$$S_t(Q) = \frac{1}{\beta_t} \mathrm{E}_Q[\beta_T S_T + \sum_{s=t+1}^{T} \beta_s \Delta D_s \,|\, \mathcal{G}_t].$$

**Theorem 15** *If transaction costs are deterministic and constant ($a_t^i = b_t^i \equiv \varepsilon$), then the set of consistent probability measures is given by*

$$\mathcal{Q} = \{Q \ll P \,|\, ||S - S(Q)||_\infty \le \varepsilon\}.$$

*We will write $\mathcal{Q}_\varepsilon$ for this set to display $\mathcal{Q}$'s dependence on $\varepsilon$.*

$\mathcal{Q}_\varepsilon$ is the perfect analogue of the set $D_\sigma$ of *consistent term structures* of chapter 2.

Theorem 15 is a special case of the following theorem.

**Theorem 16** $\mathcal{Q} = \{Q \ll P \,|\, S - b \le S(Q) \le S + a\}.$

A similar theorem in a different model was first proved by Jouini and Kallal (1995b).
Proof. Consider the trading strategies

$$\psi_s^i(\omega)(k,t,A_{t,j}) = 1_{\{i=k\}} 1_{s\in(t,T]} 1_{\omega\in A_{t,j}},$$

in words

"Buy 1 share of security $k$ at time $t$ in the event $A_{t,j}$, sell at time $T$".

$k$ runs from 1 to $m$, $t \in \{0,\ldots,T-1\}$ and $A_{t,j} \in \mathcal{G}_t$ runs through all the sets that make up the partition of $\mathcal{G}_t$. (These trading strategies form a basis $B$ of the vector space $\Phi$.) Observe that the consistency condition (26) for the strategy $\psi(k,t,A_{t,j})$ takes the form

$$\mathrm{E}_Q[\mathbf{1}_{A_{t,j}} \left\{ \beta_t(-S_t^k - b_t^k) + \beta_T S_T + \sum_{t+1}^{T} \beta \Delta D \right\}] \leq 0,$$

which can also be written as

$$S_t^k(Q) \leq S_t^k + b_t^k \quad \forall \omega \in A_{t,j}.$$

The consistency condition (26) applied to $-\psi(k,t,A_{t,j})$ is equivalent to

$$S_t^k - b_t^k \leq S_t^k(Q) \quad \forall \omega \in A_{t,j}.$$

Now the theorem can be restated as

$$\begin{aligned} & \{Q \,|\, C_0(\phi) + \mathrm{E}_Q[\textstyle\sum \beta \Delta C(\phi)] \leq 0 \,\forall \phi \in \Phi\} \\ = \; & \{Q \,|\, C_0(\phi) + \mathrm{E}_Q[\textstyle\sum \beta \Delta C(\phi)] \leq 0 \,\forall \phi \in B\} \end{aligned}$$

where $B$ denotes the basis introduced above.

One direction is trivial. To see the other direction consider the processes of cumulative buy and sell orders $(X,x)$ defined by $\Delta X_t = (\phi_{t+1} - \phi_t)^+$ and $\Delta x_t = (\phi_{t+1} - \phi_t)^-$. The relation between $\Xi = \{(X,x) \,|\, X - x \in \Phi, X \geq 0, x \geq 0, X_T - x_T = 0\}$ and $\Phi$ is not exactly one-to-one because $\Xi$ contains trading strategies that buy and sell at the same time in the same security (which generates unnecessary transaction costs). Since these trading strategies are suboptimal it does not hurt to include them and

$$\mathcal{Q} = \{Q \ll P \,|\, C_0(X,x) + \mathrm{E}_Q[\textstyle\sum \beta \Delta C(X,x)] \leq 0 \quad \forall (X,x) \in \Xi\}$$

holds. Now $C$ is linear in $(X,x)$ and the strategies $(X,x)$ constructed from $\phi(.,.,.) \in B$ and $-\phi(.,.,.) \in -B$ generate the cone $\Xi$[8]. If the consistency condition holds for any trading strategy in the generating set, then it also holds for all $(X,x) \in \Xi$. $\square$

Consider the smallest $\varepsilon$ for which $\mathcal{Q}_\varepsilon$ is not empty. This exists since $||S - S(Q)||_\infty$ is convex in $Q$.

[8] A set $B = \{y_1,\ldots,y_N\}$ generates a cone $K$ if every $x \in K$ can be written in the form $\sum_{k=1}^N \mu_k y_k$ for some nonnegative scalars $\mu_k$.

**Corollary 17** *The upper bound on transaction costs below which arbitrage is possible equals the minimax fitting error in the following sense:*

$$\min_{Q \in \mathcal{Q}} ||S - S(Q)||_{\infty}. \tag{32}$$

We would like to call an optimal $Q$ in (32) a *minimax martingale measure*.[9]

# Conclusion

Under transaction costs and trading constraints, several aspects change fundamentally compared to the classical theory. The 1-to-1 relation between price functionals and martingale measures can be maintained only if there is a numeraire without transaction costs and trading constraints. A version of the fundamental theorem of finance (the present value principle) can be maintained only of the set of admissible trading strategies is a cone and transaction costs are linear. Instead of looking for exact solutions to the present value equation (martingale measures) one should look for approximate solutions to the present value equation, with the metric depending on the structure of the transaction costs.

A theory for infinite-dimensional $\Omega$ and continuous-time stochastic processes would be nice to have but seems to be an open problem.

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[9]Unfortunately, this name was used before to denote the optimal measure in the dual of a utility maximization problem by He and Pearson (1991). Our notion of a minimax price system is independent of investors' preferences and is "minimax" in the regression sense of fitting a price functional to prices.

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